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Cosmological Models and Non-Denumerable Singularities

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Abstract

A review of the standard cosmological models shows that the positive cosmological constant gives rise to exponential increase in spatial extension. Such an increase is contradictory if extended over infinite future time, since even in an infinite universe there can only be a denumerable infinity of finite spatial units. The argument against exponential expansion is among the processes forbidden by the steady-state postulate of MacMillan.

1. *Introduction*

The empirical determination of what is the correct cosmological model has not advanced significantly in recent years (Burbidge, 1971); it seems appropriate, therefore, to employ any consistency or other extra-empirical arguments which might contribute, however indirectly, to a decision among models. One such consideration arises in connection with the cosmological constant Λ, first introduced (Einstein, 1917) as a positive quantity which is in effect a small universal anti-gravity factor. In the relativistic cosmological models with $\Lambda > 0$ the scale factor $R(t)$ increases exponentially with time; and, as we shall show, we therefore come to contradiction if existence through all time is assumed for the universe of the model.

Our argument is an extension of one that I have previously put forth (Schlegel, 1962a, b; 1965; 1967) against the Bondi–Gold–Hoyle steady-state theory. The postulated creation-of-matter process of that theory, if carried through a denumerably infinite past time, leads to a non-denumerable infinity of atoms. The implied existence of such a set, with cardinal number $\aleph$, constitutes a contradiction in the theory; for $\aleph$ is also the cardinal number of the set of mathematical points in any spatial continuum, finite or infinite (Fraenkel, 1966), and there cannot be a one–one correspondence between the set of all atoms, each of which has a finite extension, and the non-denumerable set of mathematical points. That is, the number of atoms in the universe must be at most countably infinite (Schlegel, 1962a; North, 1965). Likewise, the number of finite spatial elements cannot be non-denumerably infinite, and this limitation, we shall see, leads to the argument against the cosmological constant.

2. *The Standard Models*

By the standard relativistic models I mean those for which the time-varying spatial scale factor, $R(t)$, is described by the Friedmann equation,

$$\dot{R}^2 = C/R + \Lambda R^2/3 - k \tag{2.1}$$

where $C = (8/3)\pi\rho R^3$ is a constant, with ρ = time-varying mean energy density (gravitational constant G and speed of light c are taken as unity). We gain eighteen different models from equation (2.1), each with its own behavior for $R(t)$, depending on the values that we give to ρ, Λ, and k: ρ may be 0 or >0; Λ may be $<$, $=$, or >0; and the space-curvature index k may be $+1$, -1, or 0. Rindler (1969) explicitly discusses most of the possibilities for $R(t)$, considering the eighteen models, and I have used his results and notation in making up the following complete list (Table 1, p. 219). Further mathematical detail on many of the models is given by McVittie (1965).

3. *R(t) Increase*

It can be seen from the table that in no case with $\Lambda \leqslant 0$ do we have an exponential increase of $R(t)$. On the other hand, there is exponential R increase for every set of ρ, k values with $\Lambda > 0$; the only exception is the (b) model under $k = +1$, $\rho > 0$, which oscillates if there are the specified conditions. The positive cosmological constant has the effect, evidently, of causing space to be created. We see the increase explicitly, of course, in the line elements of the $\Lambda > 0$ models; e.g., the Robertson-Walker form of the line element for the de Sitter universe ($\Lambda > 0$, $k = 0$, $\rho = 0$) is, $R^2(t) = e^{2\alpha t}$:

$$ds^2 = dt^2 - e^{2\alpha t}[dr^2 + r^2(d\theta^2 + \sin^2\theta \cdot d\phi^2)] \tag{3.1}$$

In the B-G-H steady-state theory there is also an exponential increase in extension of space, since the line element is of the de Sitter form, but with the α of equation (3.1) explicitly replaced by Hubble's constant (there is no Λ in the theory and hence α would be 0). In discussions of the steady-state cosmology attention has usually been focused on the creation of matter, rather than of space, but there is an obvious equivalence, since the large-scale average matter density ρ must by the B-G-H principles be constant at all times. Comparing the B-G-H and standard relativistic models we can, indeed, specifically say that the matter creation tensor of the former and cosmological constant of the latter have precisely similar roles. This similarity is apparent in the basic field equations of the two theories (Bondi, 1952),

$$R_{\mu\nu} - \tfrac{1}{2}Rg_{\mu\nu} + \Lambda g_{\mu\nu} = -8\pi T_{\mu\nu} \tag{3.2}$$

and

$$R_{\mu\nu} - \tfrac{1}{2}Rg_{\mu\nu} + C_{\mu\nu} = -8\pi T_{\mu\nu} \tag{3.3}$$

wherein we see that the creation tensor $C_{\mu\nu}$ of the steady-state theory, equation (3.3), directly replaces the $\Lambda g_{\mu\nu}$ term in the general relativity theory, equation (3.2). The inconsistency which arises in the B–G–H theory

TABLE 1. $R(t)$ in the standard models. The first entry for each set of Λ, k values gives $R(t)$ for a universe with $\rho > 0$; the right-hand entry, enclosed in square brackets, is for $\rho = 0$

$\Lambda < 0$, $k = -1$: Oscillating; R_{max} is given by $3C/R^3 + 3/R^2 = -\Lambda$	[Oscillating; $R = \beta^{-1} \sin \beta t$]†
$k = 0$: Oscillating; $R^3 = (3C/-2\Lambda)(1 - \cos 3\beta t)$	[Oscillating; $R = R_0 e^{i\beta t}$]
$k = +1$: Oscillating; R_{max} is given by $3C/R^3 - 3/R^2 = -\Lambda$	[Non-physical; $R = i\beta^{-1} \cos t$]
$\Lambda = 0$, $k = -1$: $R \simeq t$ $(t \gg 0)$	[$R = t$]
$k = 0$: $R = \sqrt[3]{(9C/4)} \cdot t^{2/3}$ (Einstein–de Sitter)	[R = constant]
$k = +1$: Oscillating; $R = (1/2C)(1 - \cos \psi)$, $t = (1/2C)(\psi - \sin \psi)$, ψ an angular parameter (Friedmann)	[Non-physical; $R = it$]
$\Lambda > 0$, $k = -1$: $R \simeq R_0 e^{\alpha t}$ (R large)‡	[$R = \alpha^{-1} \sinh \alpha t$]
$k = 0$: $R \simeq R_0 e^{\alpha t}$ (R large)	[$R = R_0 e^{\alpha t}$ (de Sitter)]
$k = +1$: (a) $R \simeq R_0 e^{\alpha t}$ (R large) (Lemaitre)	[$R = \alpha^{-1} \cosh \alpha t$]
(b) Oscillating if $0 < \Lambda < \Lambda_E$§ and R initially between $R = 0$ and R at $\dot{R} = 0$ (given by $3[R - C]/R^3 = \Lambda$)	

† $\beta = \sqrt{(-\Lambda/3)}$.
‡ $\alpha = \sqrt{(\Lambda/3)}$.
§ Λ_E is the value of Λ for the Einstein static model.

with matter-creation through an achieved infinity of time is, therefore, also present in the standard models with positive Λ, in consequence of expansion of space through an infinite time.

4. *The Non-Denumerable Infinity*

We recall that a denumerable set has the cardinal number $\aleph_0$; the natural integers form such a set, and hence any set whose elements can be placed in one–one correspondence with the natural numbers is countable (denumerable), with cardinal number $\aleph_0$. However, the mathematical points in a spatial continuum constitute a set with cardinal number greater than $\aleph_0$. This larger number, which we write as $\aleph$, denotes a non-denumerable infinity: the elements of a set with multiplicity $\aleph$ cannot be counted with the unending (but lesser) infinity of the integers. Just as 2^n, for n finite, is the cardinal number of all the subsets that can be formed from n integers, so likewise $\aleph$ is the cardinal number of the set of subsets that can be formed from the set, cardinal number $\aleph_0$, of all of the positive integers. That is (Fraenkel, 1961),

$$\aleph = 2^{\aleph_0} \tag{4.1}$$

We may perhaps most simply regard equation (4.1) as defining $\aleph$ as a non-terminating product of doublings (Abian, 1965),

$$\aleph = 2^{\aleph_0} = 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot \ldots$$

The scale factor $R(t)$ determines the distance, as a function of time, between two 'points', e.g., galaxies, which are at rest in the spatial coordinate system. We have seen that for cosmological models with $\Lambda > 0$,

$$R(t) = R_0 e^{\alpha t}, \qquad (t \geqslant 0) \tag{4.2}$$

except for that $k = +1$, $\rho > 0$ model which is trapped by a particular set of initial conditions. We may regard R as the radius of some chosen volume.

By a change of base we can rewrite equation (4.2) as:

$$R = R_0 2^{(\alpha/\log_e 2)t}$$

Since we are not concerned with the numerical magnitude of $\alpha = \sqrt{(\Lambda/3)}$, except that $\Lambda > 0$, we may absorb the $\log_e 2$ factor into α, thereby giving us:

$$R = R_0 2^{\alpha t}$$

Suppose now that R_0 is expressed in some appropriate length units, as meters. Then, for time t extended forever, $t = \aleph_0$, we have, using equation (4.1),

$$R = R_0 2^{\aleph_0} = \aleph \tag{4.3}$$

The number of length units has become non-denumerably infinite.

However, we are now faced with a contradiction, since we cannot have a non-denumerable number of finite length units in even the infinite line. For length units may be enumerated successively along a line, and the cardinal number of the set of integers used in counting is of course $\aleph_0$. Or, to argue more formally, we assume that the set of unit lengths on a line does have the cardinal number $\aleph$. It must then be possible, by the property of having the same cardinal number, to set up a one–one correspondence between the unit lengths and the real numbers, which do form a set of cardinality $\aleph$. However, if this were possible we would be able to count the real numbers, because we are able to count the unit lengths of a line; but we would then contradict the established inequality, $\aleph > \aleph_0$.

The time t of our equations is the usual cosmic time. Although defined by space-like surfaces normal to the world-lines of galaxies, it is in effect the same (McVittie, 1961) as the ephemeris time of astronomy and is similarly related to the natural processes of experience. We cannot therefore expect to transform away the non-denumerability of equation (4.3), as, e.g., by transforming to a $t' = \log t$. Also, we cannot escape the contradiction by appealing to horizons existing for any one observer; the contradictory non-denumerable spatial extension applies to the entire domain of the model.

5. *Consequences and Discussion*

The inconsistency between a positive cosmological constant and an infinite future existence for the universe is an argument against any utilization of the constant. The special case of the oscillatory $\Lambda > 0$ model in which R remains finite is also discredited; for, if $\Lambda > 0$ signifies a natural process which involves a contradiction, we should not expect to find the process operating under a set of adventitious conditions. Further, if we find the positive constant to be invalid we may argue that we likewise lose the basis for any extension to a $\Lambda < 0$ constant.

It might be held that the future need not extend through $t = \aleph_0$. However, there is no evidence for processes which are leading to an eventual termination of the universe. Perhaps a better line of argument would be that the future is to be characterized by $t \to \infty$ rather than by the actual infinity $\aleph_0$. The '∞' symbol is not a transfinite number, but a prescription that n is as large a finite number as desired; and, for any finite n, 2^n is denumerable. None the less, if the universe exists for all future time it must have the property of existing for $\aleph_0$ years, since $\aleph_0$ is the cardinal number of *all* the integers. In denying that property, we would be saying there are some n values which are not included; the future, then, is terminated at some n value, indefinitely large as it might be.

Similarly, if the universe has existed forever, the number of past time units must be $\aleph_0$. And, since $\aleph_0 + 1 = \aleph_0$, with the assignment of an achieved infinite past we cannot say that the universe advances in time in any absolute sense, but only with respect to chosen events. So, if we wish to avoid a time of creation or termination (in the absence of evidence that points to their occurrence as natural processes) it seems that we must ascribe the full denumerable $\aleph_0$ infinity to the past and future time spans of the universe. Some writers have argued that an event cannot be infinitely distant in the future (or the past) because by counting one can never reach a point that is other than a finite number of years, or other units, from the present. Bertrand Russell (1938) has pointed out that such a limitation applies to any infinite class, and also, however, that the existence of such a class is in no way therefore proscribed; it is only that the infinite class must be defined by a class concept rather than by enumeration.

6. *Generalization: MacMillan's Principle*

Our result, that exponential spatial expansion through all time leads to a contradiction, is similar to the one found for the B-G-H theory, wherein creation of matter through infinite past time leads to a physical impossibility. These consequences of 'too much' space, or matter, have a similarity to the traditional Olbers' paradox problem in which emission of radiation (or of gravitational potential (Einstein, 1917)) over an infinite past gives rise, if not to contradiction, at least to conditions that are counter to what is physically observed. The processes of creation or emission lead to difficulty because they continue without reversal of direction throughout the extent of time. In an essential way they violate a postulate put forth by MacMillan (1918, 1925) in the early form of the steady-state theory which he developed (Schlegel, 1958). He proposed as a first principle of cosmology that 'The universe does not always change in any one direction', and we may dignify this statement as MacMillan's Principle.

Even for the $\Lambda = 0$ models we have a uni-directional change of R with time for $k = -1$, $R(t) \propto t$, and for $k = 0$, $R(t) \propto t^{2/3}$. So, in these models, the present $\rho > 0$ state of the universe is a highly privileged one (Tolman, 1934), of infinitesimally brief duration in contrast with the time span for which $\rho = 0$ (again, assuming that the time extent of the universe is indeed infinite). That is, if the universe has existed always in the past, the two $k \neq +1$ models predict states which are incompatible with what is now observed in the universe. MacMillan's Principle would, of course, rule out these non-oscillatory models.

There is, I judge, no justification in a broader theory for the Principle. It can be seen as a generalization of the fact that all models which change 'always in the same direction' come to difficulty, either the outright contradiction of non-denumerability or the improbability of our present observed state. Beyond its support in the instances which it generalizes, MacMillan's Principle may gain strength from our knowledge of processes throughout nature; in general we find systems of all kinds, from biological organisms to galaxies, to have natural limits of uni-directional development.

To summarize: If it is accepted that the universe exists through an actual infinity of time, the non-denumerability arguments give a firm basis for ruling out the $\Lambda > 0$ models as well as the B–G–H steady-state model. The projection of Λ to a negative value seems also then to be discredited, although we cannot say that a contradiction arises with $\Lambda < 0$. Likewise, for the remaining three realistic (non-zero density) models, $\Lambda = 0$, $\rho > 0$, MacMillan's Principle gives a plausible but yet only conjectural ground for ruling out two of them, $k = -1$ and $k = 0$. Our rather general line of argument gives support, then, only for the Friedmann ($k = +1$) oscillatory model.

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